

# OPTIMAL CONTROL OF ELLIPTIC SURFACE PDES WITH POINTWISE BOUNDS ON THE STATE

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**Abstract.** We consider a linear-quadratic optimization problem with pointwise bounds on the state for which the constraint is given by the Laplace-Beltrami equation (to have uniqueness we add an lower order term) on a two-dimensional surface . By using finite elements we approximate the optimization problem by a family of discrete problems and prove convergence rates for the discrete controls and the discrete states. Furthermore, assuming (roughly spoken) a higher regularity for the control the order of convergence improves. This extends a result known in an Euclidean setting to the surface case.

**Key words.** Linear-quadratic optimal control problem; Laplace-Beltrami equation; finite elements

**1. Introduction.** In applications the situation of a (moving) hypersurface separating two (moving) regions is a widespread setting to model various phenomena. In this general setting one may think of biological processes happening in these regions or on the interface between these regions. Examples for this scenario are cell membranes separating the environment from the cell interior, or the interface between the two phases of a two-phase flow where soluble surfactants in the bulk regions affect a certain interfacial surfactant concentration, see [6] and the references therein for a two-phase flow example.

It is a natural to consider optimization problems where the surfactant density on the surface plays the role of the state variable and to assume certain pointwise bounds for the state. To address control of the general setting above we consider in our paper a linear-quadratic PDE-constrained optimization problem on a fixed hypersurface (and not phenomena or interactions in or with the regions outside the hypersurface).

The corresponding optimization problem in an Euclidean setting is treated in [7] and we will follow the argumentation therein closely.

There are only few papers which deal with the numerics of linear-quadratic, pde constrained optimization problems on surfaces. In [9] an optimal control problem for the Laplace-Beltrami on surfaces is considered and a linear-quadratic parabolic control problem on moving surfaces is considered in [13] in the case of pointwise box constraints and in [8] in the case of pointwise bounds on the state.

Our paper is organized as follows. In Section 2 we introduce the optimization problem under consideration. Section 3 contains general material about finite elements on surfaces. Section 4 states known  $L^\infty$ -estimates which are the key ingredient in our error estimates. In Section 5 we discretize the state equation and in Section 6 the control problem. Our error estimates are formulated and proved in Section 7.

**2. The optimization problem.** Let  $S$  be a two-dimensional, closed, orientable, embedded surface in  $\mathbb{R}^3$  and  $(U, (\cdot, \cdot)_U)$  a Hilbert space. We consider the optimization

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problem

$$(2.1) \quad \begin{cases} \min_{(y,u) \in Y \times U_{ad}} J(y, u) = \frac{1}{2} \int_S |y - y_0|^2 + \frac{\alpha}{2} \|u - u_0\|_U^2 \\ \text{s.t.} \\ Ay = Bu \\ y \in Y_{ad} = \{y \in L^\infty(S) : y \leq b\}. \end{cases}$$

Here,

$$(2.2) \quad A : H^2(S) \rightarrow L^2(S), \quad Ay := -\Delta_S y + y,$$

$\alpha > 0$ ,  $y_0 \in H^1(S)$ ,  $u_0 \in U$ ,  $b \in L^\infty(S)$ ,  $R : U^* \rightarrow U$  denotes the inverse of the Frechet-Riesz isomorphism and

$$(2.3) \quad B : U \rightarrow L^2(S) \subset H^1(S)^*$$

is a linear, continuous operator. We make the assumption that

$$(2.4) \quad \exists_{u \in U_{ad}} \quad G(Bu) < b$$

where  $G = A^{-1}$ ,  $U_{ad} \subseteq U$  closed and convex and  $Y := H^1(S)$ . We have the following theorem.

**THEOREM 2.1.** *Let  $u \in U_{ad}$  denote the unique solution to (2.1). Then there exists  $\mu \in M(S)$  ( $M(S)$  denotes the set of Radon measures on  $S$ ) and  $p \in L^2(S)$  so that with  $y = G(Bu)$  there holds*

$$(2.5) \quad \int_S pAv = \int_S (y - y_0)v + \int_S v d\mu \quad \forall v \in H^2(\Omega),$$

$$(2.6) \quad (RB^*p + \alpha(u - u_0), v - u)_U \geq 0 \quad \forall v \in U_{ad},$$

and

$$(2.7) \quad \mu \geq 0, \quad y \leq b, \quad \int_S (b - y) d\mu = 0.$$

*Proof.* The proof of this theorem is along the lines of the the proof of [3, Theorem 5.2] in the Euclidean setting.  $\square$

**3. Finite Elements on Surfaces.** We triangulate  $S$  by a family  $T_h$  of flat triangles with corners (i.e. nodes) lying on  $S$ . We denote the surface of class  $C^{0,1}$  given by the union of the triangles  $\tau \in T_h$  by  $S_h$ ; the union of the corresponding nodes is denoted by  $N_h$ . Here,  $h > 0$  denotes a discretization parameter which is related to the triangulation in the following way. For  $\tau \in T$  we define the diameter  $\rho(\tau)$  of the smallest disc containing  $\tau$ , the diameter  $\sigma(\tau)$  of the largest disc contained in  $\tau$  and

$$(3.1) \quad h = \max_{\tau \in T_h} \rho(\tau), \quad \gamma_h = \min_{\tau \in T_h} \frac{\sigma(\tau)}{h}.$$

We assume that the family  $(T_h)_{h>0}$  is quasi-uniform, i.e.  $\gamma_h \geq \gamma_0 > 0$ . We let

$$(3.2) \quad X_h = \{v \in C^0(S_h) : v|_\tau \text{ linear for all } \tau \in T_h\}$$

be the space of continuous piecewise linear finite elements. Let  $N$  be a tubular neighborhood of  $S$  in which the Euclidean metric of  $N$  can be written in the coordinates  $(x^0, x) = (x^0, x^i)$  of the tubular neighborhood as

$$(3.3) \quad \bar{g}_{\alpha\beta} = (dx^0)^2 + \sigma_{ij}(x) dx^i dx^j.$$

Here,  $x^0$  denotes the globally (in  $N$ ) defined signed distance to  $S$  and  $x = (x^i)_{i=1,2}$  local coordinates for  $S$ .

For small  $h$  we can write  $S_h$  as graph (with respect to the coordinates of the tubular neighborhood) over  $S$ , i.e.

$$(3.4) \quad S_h = \text{graph } \psi = \{(x^0, x) : x^0 = \psi(x), x \in S\}$$

where  $\psi = \psi_h \in C^{0,1}(S)$  suitable. Note, that

$$(3.5) \quad |D\psi|_\sigma \leq ch, \quad |\psi| \leq ch^2.$$

The induced metric of  $S_h$  is given by

$$(3.6) \quad g_{ij}(\psi(x), x) = \frac{\partial \psi}{\partial x^i}(x) \frac{\partial \psi}{\partial x^j}(x) + \sigma_{ij}(x).$$

Hence we have for the metrics, their inverses and their determinants

$$(3.7) \quad g_{ij} = \sigma_{ij} + O(h^2), \quad g^{ij} = \sigma^{ij} + O(h^2) \quad \text{and} \quad g = \sigma + O(h^2) |\sigma_{ij} \sigma^{ij}|^{\frac{1}{2}}$$

where we use summation convention.

For a function  $f : S \rightarrow \mathbb{R}$  we define its lift  $\hat{f} : S_h \rightarrow \mathbb{R}$  to  $S_h$  by  $f(x) = \hat{f}(\psi(x), x)$ ,  $x \in S$ . For a function  $f : S_h \rightarrow \mathbb{R}$  we define its lift  $\tilde{f} : S \rightarrow \mathbb{R}$  to  $S$  by  $f = \tilde{f}$ . This terminus can be obviously extended to subsets. Let  $f \in W^{1,p}(S)$ ,  $g \in W^{1,p^*}(S)$ ,  $1 \leq p \leq \infty$  and  $p^*$  Hölder conjugate of  $p$ . In local coordinates  $x = (x^i)$  of  $S$  hold

$$(3.8) \quad \int_S \langle Df, Dg \rangle = \int_S \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \sigma^{ij}(x) \sqrt{\sigma(x)} dx^i dx^j,$$

$$(3.9) \quad \int_{S_h} \langle D\hat{f}, D\hat{g} \rangle = \int_S \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} g^{ij}(\psi(x), x) \sqrt{g(\psi(x), x)} dx^i dx^j,$$

$$(3.10) \quad \int_S \langle Df, Dg \rangle = \int_{S_h} \langle D\hat{f}, D\hat{g} \rangle + O(h^2) \|f\|_{W^{1,p}(S)} \|g\|_{W^{1,p^*}(S)},$$

and similarly,

$$(3.11) \quad \int_S f = \int_{S_h} \hat{f} + O(h^2) \|f\|_{L^1(S)}$$

where now  $f \in L^1(S)$  is sufficient.

The bracket  $\langle u, v \rangle$  denotes here the scalar product of two tangent vectors  $u, v$  (or their covariant counterparts).  $\|\cdot\|_{W^{k,p}}$  denotes the usual Sobolev norm,  $|\cdot|_{W^{k,p}} = \sum_{|\alpha|=k} \|D^\alpha \cdot\|_{L^p}$  and  $H^k = W^{k,2}$ .

**4. Some  $L^\infty$ -estimates for FE approximations.** We define

$$(4.1) \quad a : W^{1,p}(S) \times W^{1,p^*}(S) \rightarrow \mathbb{R}, \quad a(u, v) = \int_S \langle Du, Dv \rangle + uv dx,$$

$$(4.2) \quad a_h : W^{1,p}(S_h) \times W^{1,p^*}(S_h) \rightarrow \mathbb{R}, \quad a_h(u_h, v_h) = \int_{S_h} \langle Du_h, Dv_h \rangle + u_h v_h dx,$$

a discrete operator  $G_h : L^2(S) \rightarrow X_h, v \mapsto G_h v = z_h$  via

$$(4.3) \quad a_h(z_h, \varphi_h) = \int_{S_h} \hat{v} \varphi_h \quad \forall \varphi_h \in X_h$$

and have the following Lemma.

LEMMA 4.1. *Let  $v \in L^2(S)$  and  $z = Gv, z_h = G_h v$ .*

(i) *There holds*

$$(4.4) \quad \|z - \tilde{z}_h\|_{L^\infty(S)} \leq ch \|v\|.$$

(ii) *If  $v \in W^{1,s}(S)$  for some  $1 < s < 2$  then*

$$(4.5) \quad \|z - \tilde{z}_h\|_{L^\infty(S)} \leq ch^{3-\frac{2}{s}} |\log h| \|v\|_{W^{1,s}(S)}.$$

(iii) *If  $v \in L^\infty(S)$  then*

$$(4.6) \quad \|z - \tilde{z}_h\|_{L^\infty(S)} \leq ch^2 |\log h|^2 \|v\|_{L^\infty(S)}.$$

*Proof.* The proof of (i) is as in the Euclidean case and uses [5].

For  $\varphi_h \in X_h$  we define

$$(4.7) \quad F(\tilde{\varphi}_h) := a(\tilde{z}_h - z, \tilde{\varphi}_h)$$

and estimate

$$(4.8) \quad \begin{aligned} & F(\tilde{\varphi}_h) a_h(z_h, \varphi_h) + O(h^2) \|z - \tilde{z}_h\|_{W^{1,2}(S)} \|\tilde{\varphi}_h\|_{W^{1,2}(S)} \\ & \leq a_h(z_h, \varphi_h) + O(h) \|z - \tilde{z}_h\|_{W^{1,2}(S)} \|\tilde{\varphi}_h\|_{W^{1,1}(S)} \end{aligned}$$

where we used an inverse estimate. Hence  $F$  extends by Hahn-Banach theorem to an element in  $W^{-1,1}(S)$  with norm of order  $O(h^2) \|f\|_{L^2(S)}$  and then by a further application of the Hahn-Banach Theorem to an element in  $W^{-2,1}(S)$  with norm of order  $O(h^2) \|f\|_{L^2(S)}$ . A careful view shows that we are in the situation of [11, Theorem 1.2] if  $u \in W^{1,\infty}(S)$ . Hence in this case we have

$$(4.9) \quad \|z - \tilde{z}_h\|_{L^\infty(S)} \leq c \left( h |\log h| \inf_{\chi \in X_h} \|\nabla_\Gamma(z - \tilde{\chi})\|_{W^{1,\infty}(S)} + h^2 \|v\|_{L^2(S)} \right).$$

We remark that estimate (4.9) is proved in [4, Theorem 3.2].

Elliptic regularity theory and standard embedding theorems imply  $z \in W^{3,s}(S) \subset W^{2,q}(S)$ ,  $q = \frac{2s}{2-s}$ , and hence

$$(4.10) \quad \|z\|_{W^{2,q}(S)} \leq c \|z\|_{W^{3,s}(S)} \leq c \|v\|_{W^{1,s}(S)}.$$

From (4.9) and a well-known interpolation estimate we conclude

$$(4.11) \quad \|z - \tilde{z}_h\|_{L^\infty(S)} \leq ch^{2-\frac{2}{q}} |\log h| \|z\|_{W^{2,q}(S)} + ch^2 \|v\| \leq ch^{3-\frac{2}{s}} |\log h| \|v\|_{W^{1,s}(S)}$$

in view of the relation between  $s$  and  $q$ . This proves (ii).

From elliptic regularity theory we know that  $z \in W^{2,q}(S)$  for all  $1 \leq q < \infty$  with

$$(4.12) \quad \|z\|_{W^{2,q}(S)} \leq Cq \|v\|_{L^q(S)} \leq cq \|v\|_{L^\infty(S)}$$

where the constant  $C$  is independent from  $q$ . Combining this with the first inequality in (4.11) gives

$$(4.13) \quad \|z - \tilde{z}_h\|_{L^\infty(S)} \leq cq h^{2-\frac{2}{q}} |\log h| \|v\|_{L^\infty(S)}$$

so that choosing  $q = |\log h|$  proves (iii).  $\square$

**5. Finite Element Discretization of  $A$ .** In this section we adapt the argumentation from [2] to the surface case. Let  $\mu$  be a regular Borel measure in  $S$  we consider the following problem

$$(5.1) \quad Au = -\Delta_S u + u = \mu.$$

Here,  $u \in L^2(S)$  is a solution of (5.1) if

$$(5.2) \quad \int_S u A v dx = \int_S v d\mu \quad \forall v \in H^2(S).$$

Note, that  $A$  is self-adjoint.

**THEOREM 5.1.** *Let  $s \in (1, 2)$  and  $\mu \in M(S)$ . Then there exists a unique solution  $u \in W^{1,s}(S)$  of (5.1) and there holds*

$$(5.3) \quad \|u\|_{W^{1,s}(S)} \leq c(s) \|\mu\|_{M(S)}.$$

*Proof.* Let  $T : L^2(S) \rightarrow C^0(S)$  be defined by

$$(5.4) \quad A(Tf) = f, \quad f \in L^2(S).$$

$T$  is well defined in view of  $H^2(S) \subset C^0(S)$ , linear and continuous. We denote its adjoint operator by  $T^* \in L(M(S), L^2(S))$ . Then we have for all  $f \in L^2(S)$  that

$$(5.5) \quad \int_S f(T^*\mu) dx = \int_S T f d\mu$$

which implies

$$(5.6) \quad \int_S (T^*\mu) A v dx = \int_S v d\mu \quad \forall v \in H^2(S)$$

by inserting  $f = Av$  in (5.5). Hence  $u = T^*\mu$  solves (5.1). The uniqueness of the solution is obvious. To prove the regularity of  $u$  we let  $\psi \in C^0(S)$  and  $v \in H^2(S)$  be the solution of

$$(5.7) \quad Av = \psi.$$

From (5.6) we get

$$(5.8) \quad \left| \int_S u \psi dx \right| = \left| \int_S u A v dx \right| = \left| \int_S v d\mu \right| \leq \|\mu\|_{M(S)} \|v\|_{C^0(S)}.$$

By using [10, Theorem 1.4, p. 319] we deduce the existence of  $c > 0$  so that

$$(5.9) \quad \|v\|_{C^0(S)} \leq c \|\psi\|_{W^{-1,t}(S)}$$

where  $t > 2$  is arbitrary and  $c$  depends only on  $t, S$ .

From (5.8) and (5.9) we derive

$$(5.10) \quad \left| \int_S \psi u dx \right| \leq c \|\mu\|_{M(S)} \|\psi\|_{W^{-1,t}(S)}.$$

Since  $C^0(S)$  is dense in  $W^{-1,t}(S)$ ,  $\frac{1}{s} + \frac{1}{t} = 1$ , we conclude that  $u \in W^{1,s}(S)$  and (5.3).  $\square$

Let  $s \in (1, 2)$ ,  $s^*$  its Hölder conjugate and consider the bilinear form  $a$  in case  $p = s$ . We consider the following variational problem.

$$(5.11) \quad \text{Find } u \in W^{1,s}(S) \text{ so that } a(u, v) = \int_S v d\mu \quad \forall v \in W^{1,s^*}(S).$$

Note, that in view of  $s < 2$  we have  $s^* > 2$  so that  $W^{1,s^*}(S) \subset C^0(S)$ .

**THEOREM 5.2.** *Problem (5.11) has a unique solution  $u$  and  $u$  solves (5.1).*

*Proof.* Let  $u$  be the solution of (5.1). We show that  $u$  is a solution of (5.11). From Theorem 5.1 we know  $u \in W_0^{1,s}(S)$  and from (5.2) we deduce that

$$(5.12) \quad \int_S v d\mu = a(u, v) \quad \forall v \in H^2(S).$$

Hence  $u$  solves (5.1) since  $H^2(S)$  is dense in  $W^{1,s^*}(S)$ .

If  $u$  solves (5.11) then (5.12) holds and implies (5.1).  $\square$

Let  $\mu \in M(S)$  then

$$(5.13) \quad C^0(S_h) \ni u \mapsto \int_S \tilde{u} d\mu$$

is in  $(C^0(S_h))^*$ , positive and via Riesz representation theorem equal to a  $\hat{\mu} \in M(S_h)$ .

The discretization of (5.2) is given by the following problem.

$$(5.14) \quad \text{Find } u_h \in X_h \text{ so that } a_h(u_h, v_h) = \int_{S_h} v_h d\mu_h \quad \forall v_h \in X_h$$

where  $\mu_h \in B_{ch}(\hat{\mu}) \subset M(S_h)$  arbitrary but now fixed. Existence of a solution of (5.14) follows from uniqueness.

**REMARK 5.3.** *If  $\mu \in L^2(S)$  then the discretizations (5.14) and (4.3) agree for suitable  $\mu_h \in B_{ch}(\hat{\mu}) \subset M(S_h)$ .*

*Proof.* Let  $\mu = f \in L^2(\Omega)$ . The map

$$(5.15) \quad L^2(S_h) \ni v \mapsto \int_{S_h} \hat{f} v dx - \int_{S_h} v d\hat{\mu} = \int_{S_h} \hat{f} v dx - \int_S \tilde{v} f dx$$

is in  $L^2(S_h)$  with norm less or equal  $ch^2$  in view of Section 3.  $\square$

LEMMA 5.4. *Let  $v \in H^2(S)$  and  $v_h \in X_h$  the unique solution of*

$$(5.16) \quad a_h(w_h, v_h) = a(\tilde{w}_h, v) \quad \forall w_h \in X_h$$

*then*

$$(5.17) \quad \|v - \tilde{v}_h\|_{L^\infty(S)} \leq ch\|v\|_{H^2(S)}.$$

*Proof.* Let  $f = Av$  then we have in view of Section 3 that

$$(5.18) \quad a(\tilde{w}_h, v) = \int_S \tilde{w}_h f dx = \int_{S_h} w_h \hat{f} dx + O(h^2)\|w_h\|_{L^2}\|f\|_{L^2(S)} = \int_{S_h} w_h F$$

where  $F \in L^2(S_h)$  suitable and  $\|\tilde{F} - f\|_{L^2(S)} \leq O(h^2)\|f\|_{L^2(S)}$ . The claim follows as in the Euclidean setting by using the  $L^2$ -estimate from [5].  $\square$

THEOREM 5.5. *Let  $u$  be the solution of (5.1) and  $u_h$  the solution of (5.14). Then*

$$(5.19) \quad \|u - \tilde{u}_h\|_{L^2(S)} \leq c(h\|\mu\|_{M(S)} + \|\hat{\mu} - \mu_h\|_{M(S_h)}).$$

*Proof.* Let  $p \in L^2(S)$  arbitrary and  $v \in H^2(S)$  with

$$(5.20) \quad Av = p.$$

There holds

$$(5.21) \quad \begin{aligned} \int_S (u - \tilde{u}_h) p dx &= \int_S (u - \tilde{u}_h) A v dx \\ &= a(u - \tilde{u}_h, v) \\ &= \int_S v d\mu - a(\tilde{u}_h, v) \\ &= \int_S v d\mu - a_h(u_h, v_h) \\ &\leq \int_S v d\mu - \int_{S_h} v_h d\hat{\mu} + \|\hat{\mu} - \mu_h\|_{M(S_h)} \|v_h\|_{C^0(S_h)} \\ &= \int_S v d\mu - \int_S \tilde{v}_h d\mu + \|\hat{\mu} - \mu_h\|_{M(S_h)} \|v_h\|_{C^0(S_h)} \\ &\leq \|v - \tilde{v}_h\|_{C^0(S)} \|\mu\|_{M(S)} + \|\hat{\mu} - \mu_h\|_{M(S_h)} \|v_h\|_{C^0(S_h)} \\ &\leq c(h\|\mu\|_{M(S)} + \|\hat{\mu} - \mu_h\|_{M(S_h)}) \|p\|_{L^2(S)} \end{aligned}$$

where  $v_h$  as in Lemma 5.4 and we used (5.17).  $\square$

**6. Finite Element Discretization of the optimization problem.** In order to approximate problem (2.1) we consider the following family of control problems depending on the mesh parameter  $h > 0$

$$(6.1) \quad \min_{u \in U_{ad}} J_h(u) := \frac{1}{2} \int_{S_h} |y_h - \hat{y}_0|^2 + \frac{\alpha}{2} \|u - u_{0,h}\|_U^2$$

subject to

$$(6.2) \quad y_h = G_h(Bu) \quad \wedge \quad y_h(x_j) \leq b(x_j), \quad j = 1, \dots, m.$$

Here,  $u_{0,h}$  denotes an approximation to  $u_0$  with

$$(6.3) \quad \|u_0 - u_{0,h}\|_U \leq ch.$$

For every  $h > 0$  the optimization problem (6.1),(6.2) agrees with the problem which is stated in [7, (3.59)] apart from the fact that our problem is defined on  $S_h$  and the problem stated in [7, (3.59)] is defined in an open and bounded subset  $\Omega \subset \mathbb{R}^2$ . This difference does not effect the procedure how existence of an optimal solution and necessary optimality conditions are derived. Hence we get using [7, Lemma 3.2] and the definition

$$(6.4) \quad \hat{B}u = \widehat{Bu} \in L^2(S_h), \quad u \in U,$$

that the following Lemma holds.

LEMMA 6.1. *Problem (6.1) has a unique solution  $u_h \in U_{ad}$ . There exist  $\mu_1, \dots, \mu_m \in \mathbb{R}$  and  $p_h \in X_h$  so that with  $y_h = G_h(Bu_h)$  we have*

$$(6.5) \quad \begin{aligned} a_h(v_h, p_h) &= \int_{S_h} (y_h - \hat{y}_0)v_h + \sum_{j=1}^m \mu_j v_h(x_j) \quad \forall v_h \in X_h \\ (R\hat{B}^* p_h + \alpha(u_h - u_{0,h}), v - u_h)_U &\geq 0 \quad \forall v \in U_{ad}, \\ \mu_j &\geq 0, \quad y_h(x_j) \leq b(x_j), \quad j = 1, \dots, m, \text{ and } \sum_{j=1}^m \mu_j (b(x_j) - y_h(x_j)) = 0. \end{aligned}$$

We prove the following a priori bounds which are uniform in  $h$ .

LEMMA 6.2. *Let  $u_h$ ,  $\mu_j$ ,  $p_h$  and  $y_h$  as in the previous Lemma 6.1. Setting  $\mu_h = \sum_{j=1}^m \mu_j \delta_{x_j}$  by abusing notation there exists  $\bar{h} > 0$  so that*

$$(6.6) \quad \|y_h\| + \|u_h\|_U + \|\mu_h\|_{M(S_h)} \leq C \quad \text{for all } 0 < h \leq \bar{h}.$$

*Proof.* Let  $\tilde{u}$  denote an element satisfying (2.4). Since  $G(B\tilde{u})$  is continuous there exists  $\delta > 0$  so that

$$(6.7) \quad G(B\tilde{u}) \leq b - \delta \quad \text{in } S.$$

From (4.4) we deduce that there is  $h_0 > 0$  so that for all  $0 < h \leq h_0$

$$(6.8) \quad G_h(B\tilde{u}) \leq \hat{b} \quad \forall 0 < h \leq h_0$$

so that

$$(6.9) \quad J_h(u_h) \leq J_h(\tilde{u}) \quad \forall 0 < h \leq h_0$$

and hence

$$(6.10) \quad \|u_h\|_U, \|y_h\| \leq c \quad \forall 0 < h \leq h_0.$$



Let  $u$  denote the unique solution of (2.1), cf. Theorem 2.1. From (6.8) and (4.4) we infer that  $v := \frac{1}{2}u + \frac{1}{2}\tilde{u}$  satisfies

$$\begin{aligned}
(6.11) \quad \widehat{G_h(Bv)} &\leq \frac{1}{2}G(Bu) + \frac{1}{2}G(B\tilde{u}) + ch(\|Bu\| + \|B\tilde{u}\|) \\
&\leq b - \frac{\delta}{2} + ch(\|u\|_U + \|\tilde{u}\|_U) \\
&\leq b - \frac{\delta}{4}
\end{aligned}$$

for  $0 < h \leq \bar{h}$  with  $0 < \bar{h} \leq h_0$  suitable.

Since  $v \in U_{ad}$  properties (6.5), (6.10) and (6.11) imply

$$\begin{aligned}
(6.12) \quad 0 &\leq (R\hat{B}^*p_h + \alpha(u_h - u_{0,h}), v - u_h)_U \\
&= \int_{S_h} \hat{B}(v - u_h)p_h + \alpha(u_h - u_{0,h}, v - u_h)_U \\
&= a_h(G_h(Bv) - y_h, p_h) + \alpha(u_h - u_{0,h}, v - u_h)_U \\
&= \int_{S_h} (G_h(Bv) - y_h)(y_h - \hat{y}_0) + \sum_{j=1}^m \mu_j (G_h(Bv) - y_h)(x_j) \\
&\quad + \alpha(u_h - u_{0,h}, v - u_h)_U \\
&\leq C + \sum_{j=1}^m \mu_j \left( b(x_j) - \frac{\delta}{4} - y_h(x_j) \right) \\
&= C - \frac{\delta}{4} \sum_{j=1}^m \mu_j
\end{aligned}$$

where the last equality follows from (6.5). We conclude

$$(6.13) \quad \|\mu_h\|_{M(S)} \leq c$$

and the lemma is proved.  $\square$

**7. Error estimates.** In the following we assume that  $\mu_h \in B_{ch^2}(\hat{\mu}) \subset M(S_h)$  and state the following theorem.

**THEOREM 7.1.** *Let  $u$  and  $u_h$  be the solutions of (2.1) and (6.1) respectively. Then*

$$(7.1) \quad \|u - u_h\|_U + \|y - \tilde{y}_h\|_{H^1(S)} \leq ch^{\frac{1}{2}}.$$

*If in addition  $Bu \in W^{1,s}(S)$  for some  $s \in (1, 2)$  then*

$$(7.2) \quad \|u - u_h\|_U + \|y - \tilde{y}_h\|_{H^1(S)} \leq ch^{\frac{3}{2} - \frac{1}{s}} \sqrt{|\log h|}.$$

*Proof.* We test (6.5) with  $u_h$  and (2.1) with  $u$ . Adding the resulting inequalities gives

$$(7.3) \quad (R(B^*p - \hat{B}^*p_h) - \alpha(u_0 - u_{0,h}) + \alpha(u - u_h), u_h - u)_U \geq 0.$$

We recall the lift operator

$$(7.4) \quad L^2(S) \rightarrow L^2(S_h), \quad u \mapsto \hat{u}$$

and introduce its adjoint

$$(7.5) \quad L^2(S_h) \rightarrow L^2(S), \quad u \mapsto \tilde{u}$$

which is  $O(h^2)$  close to

$$(7.6) \quad L^2(S_h) \rightarrow L^2(S), \quad u \mapsto \tilde{u}.$$

There holds  $\hat{B}^* p_h = B^* \check{p}_h$  so that we conclude

$$(7.7) \quad \alpha \|u - u_h\|_U^2 \leq \int_S B(u_h - u)(p - \check{p}_h) - \alpha(u_0 - u_{0,h}, u_h - u)_U.$$

Let  $y^h = G_h(Bu) \in X_h$  and denote by  $p^h \in X_h$  the unique solution of

$$(7.8) \quad a_h(w_h, p^h) = \int_{S_h} (\hat{y} - \hat{y}_0)w_h + \int_{S_h} w_h d\mu_h \quad \forall w_h \in X_h.$$

Applying Theorem 5.5 with  $\tilde{\mu} = (y - y_0) + \mu$  we infer

$$(7.9) \quad \|p - \widetilde{p^h}\|_{L^2(S)} \leq ch(\|y - y_0\|_{L^2(S)} + \|\mu\|_{M(S)} + \|\hat{\mu} - \mu_h\|_{M(S)}).$$

We rewrite the first term on the right-hand side of (7.7)

$$(7.10) \quad \begin{aligned} & \int_S B(u_h - u)(p - \check{p}_h) \\ &= \int_S B(u_h - u)(p - \widetilde{p^h}) + \int_S B(u_h - u)(\widetilde{p^h} - \check{p}_h) \\ &= \int_S B(u_h - u)(p - \widetilde{p^h}) + \int_{S_h} B(\widehat{u_h - u})(p^h - p_h) \\ &\quad + O(h^2)\|u - u_h\|_U \|\widetilde{p^h} - \check{p}_h\|_{L^2(S)} + I_1 \\ &= \int_S B(u_h - u)(p - \widetilde{p^h}) + a_h(y_h - y^h, p^h - p_h) \\ &\quad + O(h^2)\|u - u_h\|_U \|\widetilde{p^h} - \check{p}_h\|_{L^2(S)} + I_1 \\ &= \int_S B(u_h - u)(p - \widetilde{p^h}) + \int_{S_h} (\hat{y} - y_h)(y_h - y^h) \\ &\quad + \int_{S_h} y_h - y^h d\mu_h - \sum_{j=1}^m \mu_j(y_h - y^h)(x_j) + I_1 \\ &= \int_S B(u_h - u)(p - \widetilde{p^h}) - \|\hat{y} - y_h\|_{L^2(S_h)}^2 \\ &\quad + \int_{S_h} (\hat{y} - y_h)(\hat{y} - y^h) + \int_{S_h} y_h - y^h d\mu_h + \sum_{j=1}^m \mu_j(y^h - y_h)(x_j) \\ &\quad + I_1 \end{aligned}$$

where

$$(7.11) \quad I_1 = \int_{S_h} B(\widehat{u_h - u})(p_h - \hat{p}_h)$$

and

$$(7.12) \quad |I_1| \leq O(h^2) \|p_h\|_{L^2(S_h)} \|u - u_h\|_U$$

After inserting (7.10) into (7.7) and using Young's inequality we obtain in view of (3.71), (3.55) and (3.60)

$$(7.13) \quad \begin{aligned} & \frac{\alpha}{2} \|u - u_h\|_U^2 + \frac{1}{2} \|\hat{y} - y_h\|^2 \\ & \leq c(\|p - \widetilde{p}^h\|_{L^2(S)}^2 + \|\hat{y} - y^h\|_{L^2(S_h)}^2 + \|u_0 - u_{0,h}\|_U^2) + \int_{S_h} (y_h - y^h) d\mu_h \\ & \quad + \sum_{j=1}^m \mu_j(y^h - y_h)(x_j) + |I_1|. \end{aligned}$$

We have

$$(7.14) \quad y_h - y^h \leq I_h b - \hat{b} + \hat{b} - \hat{y} + \hat{y} - y^h$$

and hence

$$(7.15) \quad \begin{aligned} \int_{S_h} y_h - y^h d\mu_h & \leq \|\mu_h\|_{M(S_h)} (\|I_h b - \hat{b}\|_{L^\infty(S_h)} + \|\hat{y} - y^h\|_{L^\infty(S_h)}) \\ & \quad + O(h^2) \|\hat{b} - \hat{y}\|_{L^\infty(S_h)} + \int_S b - y d\mu \end{aligned}$$

where the integral on the right-hand side is less or equal zero. Furthermore, we have

$$(7.16) \quad \begin{aligned} \sum_{j=1}^m \mu_j(y^h - y_h)(x_j) & = \sum_{j=1}^m \mu_j(y^h - y + y - b + b - y_h)(x_j) \\ & \leq \|y^h - \hat{y}\|_{L^\infty(S_h)} \sum_{j=1}^m \mu_j \end{aligned}$$

where we used  $y \leq b$  and  $\sum_{j=1}^m \mu_j(b - y_h)(x_j) = 0$ .

Using these estimates we can bound the right-hand side of (7.13) from above by

$$(7.17) \quad \begin{aligned} & ch^2(1 + \|y - y_0\|_{L^2(S)}^2 + \|\mu\|_{M(S)}^2 + \|u\|_{L^2(S)}^2) \\ & \quad + O(h^2) \|p_h\|_{L^2(S_h)} \|u - u_h\|_U + \|y^h - y\|_{L^\infty(S_h)}. \end{aligned}$$

Testing (6.5) with  $p_h$  yields

$$(7.18) \quad \|p_h\|_{L^2(S_h)}^2 \leq c \|p_h\|_{L^2(S_h)} + \|p_h\|_{L^\infty(S_h)} \leq ch^{-1} \|p_h\|_{L^2(S_h)}$$

where we used for the last inequality an inverse inequality. We conclude

$$(7.19) \quad \|p_h\|_{L^2(S_h)} \leq ch^{-1}.$$

Putting facts together shows that the right-hand side of (7.13) can be bounded from above by

$$(7.20) \quad ch^2 + \|y^h - \hat{y}\|_{L^\infty(S_h)}.$$

The norm in (7.20) can be estimated by  $ch\|u\|_{L^2(S)}$  by using (4.4) or by

$$(7.21) \quad ch^{3-\frac{2}{s}}|\log h|\|u\|_{L^2(S)}$$

by using Lemma 4.1 depending on the assumption on  $Bu$ .  $\square$

**COROLLARY 7.2.** *Let  $u$  and  $u_h$  be as in Theorem 7.1 (i) and assume that  $Bu, Bu_h \in L^\infty(S)$  are uniformly bounded in the  $L^\infty$ -norm. Then, for  $h$  small enough*

$$(7.22) \quad \|u - u_h\|_U + \|y - y_h\|_{H^1} \leq ch|\log h|.$$

*Proof.* We set  $\bar{y} = GBu_h$  and rewrite the first summand on the right-hand side of (7.7) as follows

$$(7.23) \quad \begin{aligned} \int_S B(u_h - u)(p - \check{p}_h) &= \int_S pA(\bar{y} - y) - \int_{S_h} \widehat{B(u_h - u)} \hat{p}_h + O(h^2)\|u_h - u\|_U \|\check{p}_h\|_{L^2(S)} \\ &= \int_S pA(\bar{y} - y) - a_h(y_h - y^h, p_h) + O(h^2)\|u_h - u\|_U \|\check{p}_h\|_{L^2(S)} \\ &\stackrel{(2.5), (6.5)}{=} \int_S (y - y_0)(\bar{y} - y) + \int_S \bar{y} - y d\mu \\ &\quad - \int_{S_h} (y_h - \hat{y}_0)(y_h - y^h) - \sum_{j=1}^m \mu_j(y_h - y^h)(x_j) \\ &\quad + O(h^2)\|u_h - u\|_U \|\check{p}_h\|_{L^2(S)}. \end{aligned}$$

We rewrite the sum of the first and the third summand on the right-hand side as

$$(7.24) \quad \begin{aligned} &\int_S (y - \tilde{y}_h + \tilde{y}_h - y_0)(\bar{y} - y) - \int_{S_h} (y_h - \hat{y}_0)(y_h - y^h) \\ &= \int_S (y - \tilde{y}_h)(\bar{y} - \tilde{y}_h + \tilde{y}_h - y) + O(h^2)\|\tilde{y}_h - y_0\|_{L^2(S)}\|\bar{y} - y\|_{L^2(S)} \\ &\quad + \int_{S_h} (y_h - \hat{y}_0)(\hat{y} - y_h + y^h - \hat{y}) \\ &\leq -\|y - \tilde{y}_h\|_{L^2(S)} + \|y_h - \hat{y}_0\|_{L^2(S_h)}(\|\hat{y} - y_h\|_{L^2(S_h)} + \|y^h - \hat{y}\|_{L^2(S_h)}) \\ &\quad + O(h^2)\|\tilde{y}_h - y_0\|_{L^2(S)}\|u - u_h\|_U. \end{aligned}$$

We use

$$(7.25) \quad \bar{y} - y \leq \bar{y} - \tilde{y}_h + \tilde{y}_h - \widetilde{I_h b} + \widetilde{I_h b} - b + b - y$$

and

$$(7.26) \quad y_h \leq I_h b \quad \wedge \quad \int_S b - y d\mu = 0$$

so that

$$(7.27) \quad \int_S \bar{y} - y d\mu \leq c\|\mu\|_{M(S)} \left\{ \|\bar{y} - \tilde{y}_h\|_{L^\infty(S)} + \|\widetilde{I_h b} - b\|_{L^\infty(S)} \right\}.$$

Using (7.16) and putting facts together leads to

$$\begin{aligned}
(7.28) \quad & \|u - u_h\|_U^2 + \|y - \tilde{y}_h\|_{L^2(S)}^2 \leq \|u_0 - u_{0,h}\|_U^2 \\
& + \|\tilde{y}_h - y_0\|_{L^2(S)} (\|\hat{\tilde{y}} - y_h\|_{L^2(S_h)} + \|y^h - \hat{y}\|_{L^2(S_h)}) \\
& + O(h^2) \|u - u_h\|_U \\
& + c(\|\bar{y} - \tilde{y}_h\|_{L^\infty(S)} + \|\widetilde{I_h b} - b\|_{L^\infty(S)} + \|y^h - \hat{y}\|_{L^\infty(S_h)}) \\
& + O(h^2) \|u_h - u\|_U \|\tilde{p}_h\|_{L^2(S)}
\end{aligned}$$

Using Lemma 6.2, Lemma 3.1 (iii) and (7.18) then yields

$$(7.29) \quad \|u - u_h\|_U^2 + \|y - \tilde{y}_h\|_{L^2(S)}^2 \leq c(h^2 + h^2 |\log h|^2)$$

so that the claim follows.  $\square$

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